

ON THE SEVERI TYPE INEQUALITIES FOR IRREGULAR SURFACES

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ABSTRACT. Let X be a minimal surface of general type and maximal Albanese dimension with irregularity $q \geq 2$. We show that $K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q-2)$ if $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$, and also obtain the characterization of the equality. As a consequence, we prove a conjecture of Manetti on the geography of irregular surfaces if $K_X^2 \geq 36(q-2)$ or $\chi(\mathcal{O}_X) \geq 8(q-2)$, and we also prove a conjecture that surfaces of general type and maximal Albanese dimension with $K_X^2 = 4\chi(\mathcal{O}_X)$ are exactly the resolution of double covers of abelian surfaces branched over ample divisors with at worst simple singularities.

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1. INTRODUCTION

Let X be a minimal surface of general type and maximal Albanese dimension with irregularity $q \geq 2$. The Severi inequality states that

$$K_X^2 \geq 4\chi(\mathcal{O}_X).$$

It was proved by Manetti [Man03] using the analysis on the one-forms on X provided that K_X is ample; and it was completely proved by Pardini [Par05] based on Albanese lifting technique and limiting argument. A natural question arising is when the equality holds.

Conjecture 1.1 ([Man03, § 0] & [MLP12, § 5.2]). *Let X be a minimal surface of general type and maximal Albanese dimension. Then $K_X^2 = 4\chi(\mathcal{O}_X)$ if and only if the canonical model of X is a flat double cover of an abelian surface branched over an ample divisor with at worst simple singularities.*

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Here we recall that the branch divisor of a double cover is said to have at worst simple singularities if the multiplicities of the singularities appearing in the process of the canonical resolution are at most three (see Definition 4.3). The above conjecture was confirmed by Manetti in [Man03] under the assumption that K_X is ample.

The next task in the study of the geography on irregular surfaces of general type and maximal Albanese dimension is how the irregularity q influences the relation between K_X^2 and $\chi(\mathcal{O}_X)$. As a first attempt in this direction, Mendes Lopes and Pardini proved in [MLP11] that

$$K_X^2 \geq 4\chi(\mathcal{O}_X) + \frac{10}{3}q - 8, \quad \text{if } K_X \text{ is ample and } q \geq 5.$$

In [Man03, Conjecture 7.5], Manetti suggested the following inequality

Conjecture 1.2 (Manetti). *Let X be a minimal surface of general type and maximal Albanese dimension with $q \geq 4$. Then*

$$K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q - 3). \quad (1-1)$$

Our main result is the following.

Theorem 1.3. *Let X be a minimal surface of general type and maximal Albanese dimension. Then*

(1)

$$K_X^2 \geq \min \left\{ \frac{9}{2}\chi(\mathcal{O}_X), 4\chi(\mathcal{O}_X) + 4(q - 2) \right\}. \quad (1-2)$$

(2) *Assume that $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$. Then*

$$K_X^2 = 4\chi(\mathcal{O}_X) + 4(q - 2), \quad (1-3)$$

if and only if the canonical model of X is a flat double cover of Y whose branch divisor R has at worst simple singularities and $K_Y \cdot R = 4(q - 2)$, where

$$\begin{cases} Y \text{ is an abelian surface,} & \text{if } q = 2; \\ Y \cong E \times C \text{ with } g(E) = 1 \text{ and } g(C) = q - 1, & \text{if } q \geq 3. \end{cases}$$

Based on the above result, we obtain a positive answer to Conjecture 1.1 and a partial support of Conjecture 1.2.

Corollary 1.4. (1) *Conjecture 1.1 is true.*

(2) *If X is a minimal surface of general type and maximal Albanese dimension with $K_X^2 \geq 36(q - 2)$ or $\chi(\mathcal{O}_X) \geq 8(q - 2)$, then*

$$K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q - 2). \quad (1-4)$$

In particular, (1-1) holds in this case. Moreover if $K_X^2 > 36(q - 2)$ or $\chi(\mathcal{O}_X) > 8(q - 2)$, then the equality in (1-4) can hold if and only if the canonical model of X is a flat double cover of Y whose branch divisor R has at worst simple singularities and $K_Y \cdot R = 4(q - 2)$, where Y is the same as in Theorem 1.3(2).

The main idea of the proof is as follows.

- Step 1 We apply Pardini's Albanese lifting technique in [Par05] verbatim to construct a sequence of fibred surfaces $f_n : \tilde{X}_n \rightarrow \mathbb{P}^1$ such that the slope λ_{f_n} converges to $K_X^2/\chi(\mathcal{O}_X)$ when n tends to infinity. With the help of [LZ15, Theorem 3.1], the proof of Theorem 1.3 is then reduced to the case where f_n 's are all double cover fibrations (cf. [LZ15, Definition 1.4]) for n sufficiently large.
- Step 2 We make use of Xiao's linear bound on the order of the automorphism group of surfaces of general type (cf. [Xia94]) to show that the involution on \tilde{X}_n induces a well-defined involution σ on X provided that $n = p$ is prime and large enough. Moreover, it is shown that the Albanese map of X factors through $X/\langle\sigma\rangle$.
- Step 3 We investigate irregular surfaces of general type and maximal Albanese dimension whose Albanese map factors through a double cover, and then apply induction on the degree of the Albanese map to finish the proof of Theorem 1.3.

The above three steps are completed in sections 2, 3 and 4 respectively. One of the main advantages of Pardini's Albanese lifting technique, compared to those in [Man03, MLP11], is that the canonical divisor of X does not have to be ample. This allows us to deal with a large class of surfaces. We remark that the high dimensional Severi inequality has been recently obtained independently by Barja [Bar15] and Zhang [Zha14], where Pardini's Albanese lifting technique is also crucial to both of their proofs.

Notations and conventions: We work over the complex number. The surface X is a minimal smooth surface of general type and maximal Albanese dimension with irregularity $q \geq 2$. The canonical divisor of X is denoted by K_X , and $\chi(\mathcal{O}_X)$ is referred to be the characteristic of the structure sheaf of X . The Albanese variety of X is denoted by $\text{Alb}(X)$ and $\text{Alb}_X : X \rightarrow \text{Alb}(X)$ is the Albanese map. The surface X is said to be of maximal Albanese dimension if the dimension of the image $\text{Alb}_X(X)$ is 2.

Acknowledgements: When our paper was prepared, Barja kindly informed us that Conjecture 1.1 was also proved in his joint work [BPS15] with Pardini and Stoppino using different techniques.

2. CONSTRUCTION OF FIBRED SURFACES

In the famous paper [Par05], Pardini introduced a clever Albanese lifting technique to construct a sequence of fibred surfaces $f_n : \tilde{X}_n \rightarrow \mathbb{P}^1$ such that the slope λ_{f_n} converges to $K_X^2/\chi(\mathcal{O}_X)$ when n tends to infinity. For readers' convenience, we recall a brief formulation of the construction.

Let $\text{Alb}_X : X \rightarrow \text{Alb}(X)$ be the Albanese map of X . Let L be a very ample line bundle on $\text{Alb}(X)$ and $H = (\text{Alb}_X)^*(L)$. For any integer $n \geq 2$, let $\mu_n : \text{Alb}(X) \rightarrow \text{Alb}(X)$ be the multiplication by n and consider the following Cartesian diagram:

$$\begin{array}{ccc} X_n & \xrightarrow{\nu_n} & X \\ a_n \downarrow & & \downarrow \text{Alb}_X \\ \text{Alb}(X) & \xrightarrow{\mu_n} & \text{Alb}(X) \end{array}$$

Then X_n is a smooth surface of maximal Albanese dimension with the following invariants

$$K_{X_n}^2 = n^{2q} K_X^2, \quad \chi(\mathcal{O}_{X_n}) = n^{2q} \chi(\mathcal{O}_X).$$

According to [LB92, § 2, Proposition 3.5], we have the following numerical equivalence on $\text{Alb}(X)$:

$$\mu_n^*(L) \sim_{\text{num}} n^2 L.$$

It follows that

$$H_n := a_n^*(L) \sim_{\text{num}} n^{-2} \nu_n^*(H).$$

Note that $\deg \nu_n = \deg \mu_n = n^{2q}$. Hence

$$H_n^2 = n^{2q-4} H^2, \quad K_{X_n} \cdot H_n = n^{2q-2} K_X \cdot H.$$

Now let $D_1, D_2 \in |H_n|$ be two general smooth curves, $C_1 = D_1 + D_2$ and $C_2 \in |2H_n|$ be a general curve, such that C_2 is smooth and C_1 and C_2 intersect transversely at $(2H_n)^2$ points. Let $\tilde{X}_n \rightarrow X_n$ be the blow up of the intersection points of C_1 and C_2 , so that we get a (relatively minimal) fibration $f_n : \tilde{X}_n \rightarrow \mathbb{P}^1$ of genus g_n with the following invariants:

$$\begin{aligned} g_n &= 1 + n^{2q-2} K_X \cdot H + 2n^{2q-4} H^2, \\ K_{f_n}^2 &= n^{2q} K_X^2 + 8n^{2q-2} K_X \cdot H + 12n^{2q-4} H^2, \\ \chi_{f_n} &= n^{2q} \chi(\mathcal{O}_X) + n^{2q-2} K_X \cdot H + 2n^{2q-4} H^2. \end{aligned}$$

Hence we obtain a sequence of fibred surfaces $f_n : \tilde{X}_n \rightarrow \mathbb{P}^1$ with slopes

$$\lambda_{f_n} = \frac{K_{f_n}^2}{\chi_{f_n}} = \frac{K_X^2 + 8n^{-2} K_X \cdot H + 12n^{-4} H^2}{\chi(\mathcal{O}_X) + n^{-2} K_X \cdot H + 2n^{-4} H^2} \longrightarrow \frac{K_X^2}{\chi(\mathcal{O}_X)}, \quad \text{as } n \longrightarrow +\infty.$$

Theorem 2.1. *Let f_n 's be the fibrations constructed as above. Then either*

$$K_X^2 \geq \frac{9}{2} \chi(\mathcal{O}_X), \quad (2-1)$$

or the fibration f_n is a double cover fibration (cf. [LZ15, Definition 1.4]) for any sufficiently large n .

Proof. Assume that $K_X^2 < \frac{9}{2} \chi(\mathcal{O}_X)$, i.e., $K_X^2 \leq \frac{9}{2} \chi(\mathcal{O}_X) - \frac{1}{2}$. We have to show that f_n is a double cover fibration if n is sufficiently large.

As

$$\lambda_{f_n} \longrightarrow \frac{K_X^2}{\chi(\mathcal{O}_X)}, \quad \text{as } n \longrightarrow +\infty,$$

there exists N_1 such that

$$\lambda_{f_n} \leq \frac{K_X^2}{\chi(\mathcal{O}_X)} + \frac{1}{8\chi(\mathcal{O}_X)} \leq \frac{9}{2} - \frac{3}{8\chi(\mathcal{O}_X)}, \quad \forall n \geq N_1. \quad (2-2)$$

On the other hand, note that

$$\frac{18(g_n - 1)}{4g_n + 3} = \frac{18(K_X \cdot H + 2n^{-2} H^2)}{4K_X \cdot H + 8n^{-2} H^2 + 7n^{2-2q}} \longrightarrow \frac{9}{2}, \quad \text{as } n \longrightarrow +\infty.$$

Hence there exists N_2 such that

$$\frac{18(g_n - 1)}{4g_n + 3} \geq \frac{9}{2} - \frac{1}{4\chi(\mathcal{O}_X)}, \quad \forall n \geq N_2.$$

According to [LZ15, Theorem 3.1(ii)], it follows that f_n must be a double cover fibration once $n \geq \max\{N_1, N_2\}$. The proof is complete. \square

3. FACTORIZATION OF THE ALBANESE MAP

The main purpose of this section is to prove the following theorem on the factorization of the Albanese map.

Theorem 3.1. *Let X be a surface of maximal Albanese dimension with $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$. Then there exists an involution σ on X such that the Albanese map Alb_X factors through a double cover $\pi : X \rightarrow Y := X/\langle\sigma\rangle$, i.e., the following diagram commutes:*

$$\begin{array}{ccccc} X & \xrightarrow{\pi} & Y = X/\langle\sigma\rangle & \xrightarrow{\quad} & \text{Alb}_X(X) \\ & \searrow & \text{Alb}_X & \nearrow & \\ & & & & \end{array}$$

In order to prove Theorem 3.1, we may assume that $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$ in this section. Hence by Theorem 2.1, the fibration f_n constructed in the last section is a double cover fibration for any sufficiently large n , i.e., there exists an involution $\tilde{\sigma}_n$ on \tilde{X}_n such that $f_n \circ \tilde{\sigma}_n = f_n$. We always assume that n is large enough in this section such that f_n is a double cover fibration. Remark however that there might be more than one involution on \tilde{X}_n , in which case we choose and fix one. Since X_n is the minimal model of \tilde{X}_n , $\tilde{\sigma}_n$ induces an involution σ_n on X_n . First we have the following easy lemma.

Lemma 3.2. *Let $a_n : X_n \rightarrow \text{Alb}(X)$ be the map constructed in section 2. Then $a_n \circ \sigma_n = a_n$.*

Proof. Let $x \in X_n$ be a general point, $x' = \sigma_n(x)$, $y = a_n(x)$, and $y' = a_n(x')$. Then it suffices to prove $y = y'$. Suppose on the contrary $y \neq y'$. Let Λ be the pencil generated by $C_1 = D_1 + D_2$ and C_2 in section 2. Since L is very ample on $\text{Alb}(X)$, the complete linear system $|L|$ (hence also $|2L|$) separates the two points y and y' , from which it follows that $|H_n|$ (hence also $|2H_n|$) separates the two points x and x' . According to the generality of D_1, D_2 and C_2 in the construction, one obtains that Λ also separates the two points x and x' , which means that $x' \notin C_x$ and $x \notin C_{x'}$, where C_x (resp. $C_{x'}$) is the member of Λ containing x (resp. x'). Because $f_n \circ \tilde{\sigma}_n = f_n$, it follows that $\sigma_n(C) = C$ for any member $C \in \Lambda$. In particular, $\sigma_n(C_x) = C_x$, from which it follows that $x' = \sigma_n(x) \in \sigma_n(C_x) = C_x$. This gives a contradiction. The proof is complete. \square

The next lemma is key to the proof of Theorem 3.1, which relies highly on Xiao's linear bound on the order of the automorphism group of surfaces of general type (cf. [Xia94]).

Lemma 3.3. *If $n = p$ is a prime number and sufficiently large, then σ_p induces a well-defined involution σ on X , i.e., one has the following commutative diagram*

$$\begin{array}{ccc} X_p & \xrightarrow{\nu_p} & X \\ \sigma_p \downarrow & & \downarrow \sigma \\ X_p & \xrightarrow{\nu_p} & X \end{array}$$

Proof. Let $\mu_n : \text{Alb}(X) \rightarrow \text{Alb}(X)$ be the multiplication by n on the abelian variety $\text{Alb}(X)$ as in section 2. Then μ_n is an abelian cover with Galois group $G_n \cong (\mathbb{Z}/n\mathbb{Z})^{\oplus 2q}$. By the construction of X_n in section 2, it is a fibre-product. Hence $\nu_n : X_n \rightarrow X$ is also an abelian cover with Galois group $G_n \cong (\mathbb{Z}/n\mathbb{Z})^{\oplus 2q}$. If

$$\sigma_n \circ \tau \circ \sigma_n^{-1} \in G_n, \quad \forall \tau \in G_n, \quad (3-1)$$

then one defines $\sigma(x) = \nu_n(\sigma_n(x_n))$, where $x_n \in X_n$ is any point satisfying $\nu_n(x_n) = x$. By (3-1), σ is well-defined and the above diagram is obviously commutative. Hence it suffices to prove (3-1) when $n = p$ is a prime number and sufficiently large.

Let \tilde{G}_p be the subgroup of the automorphism group $\text{Aut}(X_p)$ of X_p generated by G_p and σ_p . Let p^d be the order of any Sylow p -subgroup of \tilde{G}_p . Note that $K_{X_p}^2 = p^{2q} K_X^2$, and according to [Xia94, Theorem 1], there exists a universal coefficient c such that

$$p^d \leq |\tilde{G}_p| \leq |\text{Aut}(X_p)| \leq cK_{X_p}^2 = p^{2q} \cdot cK_X^2.$$

Hence $d \leq 2q$ if $p > cK_X^2$. On the other hand, it is clear that $d \geq 2q$ due to the fact that $|G_p| = p^{2q}$. Therefore G_p is a Sylow p -subgroup of \tilde{G}_p if $p > cK_X^2$. Note that (3-1) is equivalent to

$$\sigma_n \circ G_n \circ \sigma_n^{-1} = G_n \text{ as subgroups of } \tilde{G}_n, \quad (3-2)$$

Assume that $\sigma_n \circ G_n \circ \sigma_n^{-1} \neq G_n$, then the number of Sylow p -subgroups of \tilde{G}_p is $n_p \geq 2$. On the other hand, according to Sylow's theorems, it is known that n_p divides $|\tilde{G}_p|/p^{2q}$, and

$$n_p \equiv 1 \pmod{p}.$$

It follows that $n_p \geq p + 1$, and

$$p^{2q} \cdot cK_X^2 \geq |\text{Aut}(X_p)| \geq |\tilde{G}_p| \geq n_p \cdot p^{2q} \geq p^{2q}(p + 1).$$

It is a contradiction if $p > cK_X^2$. Hence if $p > cK_X^2$, then (3-2) holds and hence (3-1) holds too. The proof is complete. \square

Proof of Theorem 3.1. The existence of an involution follows from Lemma 3.3. Let σ be as in Lemma 3.3. Then according to Lemma 3.2,

$$\text{Alb}_X \circ \sigma \circ \nu_p = \text{Alb}_X \circ \nu_p \circ \sigma_p = \mu_p \circ a_p \circ \sigma_p = \mu_p \circ a_p = \text{Alb}_X \circ \nu_p.$$

Since $\nu_p : X_p \rightarrow X$ is surjective, it follows that $\text{Alb}_X \circ \sigma = \text{Alb}_X$, i.e., the Albanese map Alb_X factors through the double cover $\pi : X \rightarrow Y := X/\langle \sigma \rangle$. \square

4. IRREGULAR SURFACES COMING FROM DOUBLE COVERS

Similar to the last section, we will always assume that $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$ in this section unless other explicit statements. According to Theorem 3.1, the Albanese map Alb_X factors through a double cover $\pi : X \rightarrow Y := X/\langle \sigma \rangle$. In the section, we deal with the irregular surfaces coming from double covers.

Let $Y' \rightarrow Y$ be the resolution of singularities, and Y_0 be a minimal model of Y' . The double cover π induces a double cover

$$\pi_0 : S_0 \longrightarrow Y_0, \quad (4-1)$$

and X is nothing but the minimal smooth model of S_0 . We want to compute the invariants of X according to the double cover π_0 . First we have the following lemma

Lemma 4.1. *Let $\text{Alb}_{Y'} : Y' \rightarrow \text{Alb}(Y')$ be the Albanese map of Y' . Then $\text{Alb}(X) \cong \text{Alb}(Y')$. Hence Y' (and Y_0) is also a smooth surface of maximal Albanese dimension with $q(Y') = q$.*

Proof. Since Y' is the resolution of singularities of Y , by a possible blow up: $\tilde{X} \rightarrow X$, one gets a generically finite map $\tilde{X} \rightarrow Y'$ of degree two induced by π . Therefore, we obtain a surjective map $\alpha : \text{Alb}(X) = \text{Alb}(\tilde{X}) \rightarrow \text{Alb}(Y')$. On the other hand, one has the composition map $Y' \rightarrow Y \rightarrow \text{Alb}(X)$ whose image is $\text{Alb}_X(X)$. By the universal property of the Albanese variety, one obtains a surjective map $\beta : \text{Alb}(Y') \rightarrow \text{Alb}(X)$. It is easy to check that α and β are inverse to each other. The proof is complete. \square

According to the classification of algebraic surfaces, the Kodaira dimension $\kappa(Y_0) = \kappa(Y') \geq 0$. More precisely, we have the following possibilities for the minimal model Y_0 :

$$\begin{cases} \kappa(Y_0) = 0 : & Y_0 \text{ is an abelian surface, in which case } q = 2; \\ \kappa(Y_0) = 1 : & Y_0 \cong E \times C \text{ with } g(E) = 1 \text{ and } g(C) = q - 1, \text{ in which case } q \geq 3; \\ \kappa(Y_0) = 2 : & Y_0 \text{ is a minimal surface of general type with } q(Y_0) = q. \end{cases}$$

Proposition 4.2. *If $\kappa(Y_0) = 0$ or 1, then*

$$K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q - 2). \quad (4-2)$$

Moreover, the equality in (4-2) holds if and only if the canonical model of X is isomorphic to S_0 , the branch divisor R_0 of the double cover π_0 has at worst simple singularities, and $K_{Y_0} \cdot R_0 = 4(q - 2)$.

Proof. Let π_0 be the induced double cover as in (4-1) whose cover datum is

$$\mathcal{O}_{Y_0}(R_0) \equiv \mathcal{L}_0^{\otimes 2}.$$

The surface S_0 may not be smooth. To get the smooth model, we perform the canonical resolution as follows.

$$\begin{array}{ccccccc} S_t & \xrightarrow{\phi_t} & S_{t-1} & \xrightarrow{\phi_{t-1}} & \cdots & \xrightarrow{\phi_2} & S_1 & \xrightarrow{\phi_1} & S_0 \\ \tilde{\pi} = \pi_t \downarrow & & \downarrow \pi_{t-1} & & & & \downarrow \pi_1 & & \downarrow \pi_0 \\ Y_t & \xrightarrow{\psi_t} & Y_{t-1} & \xrightarrow{\psi_{t-1}} & \cdots & \xrightarrow{\psi_2} & Y_1 & \xrightarrow{\psi_1} & Y_0 \end{array}$$

where S_t is smooth and ψ_i 's are successive blowing-ups resolving the singularities of R_0 ; the map $\phi_i : X_i \rightarrow Y_i$ is the double cover determined by

$$\mathcal{O}_{Y_i}(R_i) \equiv \mathcal{L}_i^{\otimes 2}$$

with

$$R_i = \psi_i^*(R_{i-1}) - 2m_{i-1}E_i, \quad \mathcal{L}_i = \psi_i^*(\mathcal{L}_{i-1}) \otimes \mathcal{O}_{Y_i}(-m_{i-1}E_i),$$

where E_i is the exceptional divisor of ψ_i , d_i is the multiplicity of the singular point y_i in R_i and $m_i = [d_i/2]$, ($[]$ stands for the integral part).

Definition 4.3. If $d_i = 2$ or 3 for any $0 \leq i \leq t-1$, then we say that R_0 has at worst simple singularities. In some literature (eg. [Xia91]), such singularities are also called negligible singularities.

The invariants of S_t is computed by the following formulas (cf. [BHPV04, § V.22]):

$$K_{S_t}^2 = 2K_{Y_0}^2 + 2K_{Y_0} \cdot R_0 + \frac{1}{2}R_0^2 - 2 \sum_{i=0}^{t-1} (m_i - 1)^2; \quad (4-3)$$

$$\chi(\mathcal{O}_{S_t}) = 2\chi(\mathcal{O}_{Y_0}) + \frac{1}{4}K_{Y_0} \cdot R_0 + \frac{1}{8}R_0^2 - \sum_{i=0}^{t-1} \frac{1}{2}m_i(m_i - 1). \quad (4-4)$$

If $Y_0 \cong E \times C$ with $g(E) = 1$ and $g(C) = q - 1$, then one has $H \cdot R_0 \geq 2$ since X is of general type, where $H = \{p\} \times C \subseteq Y_0$ for any $p \in E$. Hence

$$K_{Y_0} \cdot R_0 = (2g(C) - 2)H \cdot R_0 \geq 4(q - 2). \quad (4-5)$$

If Y_0 is an abelian surface, then it is clear that $K_{Y_0} \cdot R_0 = 0 = 4(q - 2)$.

Note that X is nothing but the minimal model of S_t , which implies that $K_X^2 \geq K_{S_t}^2$ and $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{S_t})$. Hence

$$\begin{aligned} K_X^2 - 4\chi(\mathcal{O}_X) &\geq K_{S_t}^2 - 4\chi(\mathcal{O}_{S_t}) \\ &= 2(K_{Y_0}^2 - 4\chi(\mathcal{O}_{Y_0})) + K_{Y_0} \cdot R_0 + 2 \sum_{i=0}^{t-1} (m_i - 1) \\ &\geq K_{Y_0} \cdot R_0 \geq 4(q - 2). \end{aligned}$$

To characterize the equality, first it is clear that the equality holds if the canonical model of X is isomorphic to S_0 and the equality in (4-5) holds. Conversely, if the equality in (4-2) holds, then $X = S_t$, the equality in (4-5) holds, and $m_i = 1$ for any $0 \leq i \leq t-1$ according to the above arguments. The later implies that $d_i = 2$ or 3 , i.e., R_0 has at worst simple singularities, from which it follows that the inverse image of these exceptional curves is a union of (-2) -curves in $X = S_t$. This together with the fact that Y_0 contains no rational curves shows that S_0 is just the canonical model of X . \square

Lemma 4.4. Assume that Y_0 is of general type.

(1) if $K_{Y_0}^2 \geq 4\chi(\mathcal{O}_{Y_0}) + 4(q(Y_0) - 2)$, then

$$K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q - 2), \quad (4-6)$$

and the equality can hold only when $q = 2$, $\pi = \pi_0$ is unramified and $K_Y^2 = 4\chi(\mathcal{O}_Y)$;

(2) if $K_{Y_0}^2 \geq \frac{9}{2}\chi(\mathcal{O}_{Y_0})$, then

$$K_X^2 > 4\chi(\mathcal{O}_X) + 4(q - 2). \quad (4-7)$$

Proof. We follow the notations introduced in Proposition 4.2.

(1) If $K_{Y_0}^2 \geq 4\chi(\mathcal{O}_{Y_0}) + 4(q(Y_0) - 2)$, then by (4-3) and (4-4),

$$\begin{aligned}
K_X^2 - 4\chi(\mathcal{O}_X) &\geq K_{S_t}^2 - 4\chi(\mathcal{O}_{S_t}) \\
&= 2(K_{Y_0}^2 - 4\chi(\mathcal{O}_{Y_0})) + K_{Y_0} \cdot R_0 + 2 \sum_{i=0}^{t-1} (m_i - 1) \\
&\geq 2(K_{Y_0}^2 - 4\chi(\mathcal{O}_{Y_0})) \\
&\geq 8(q(Y_0) - 2) = 8(q - 2) \geq 4(q - 2).
\end{aligned}$$

This proves (4-6). If the equality in (4-6) holds, then $q = 2$, $K_{Y_0}^2 = 4\chi(\mathcal{O}_{Y_0})$, $X = S_t$ is minimal, $m_i = 1$ for any $0 \leq i \leq t-1$ and $K_{Y_0} \cdot R_0 = 0$. The last implies that $R_0 = \emptyset$ or R_0 is a union of (-2) -curves since Y_0 is minimal of general type. If $R_0 = \emptyset$, then $\pi = \pi_0$ is unramified and hence $Y = Y_0$ as required. Hence it suffices to deduce a contradiction if R_0 is a union of (-2) -curves.

Let $D = \sum_{i=1}^l C_i \subseteq R_0$ be a connected component. First we claim that D has at worst simple double points as its singularities; otherwise, let p be a singularity of D which is not a simple double point, and $D' = \sum_{i=1}^k C_i \subseteq D$ be all the irreducible components of D through p . Then either $k \geq 3$ and $C_i \cdot C_j \geq 1$ for any $1 \leq i < j \leq k$, or $k = 2$ and $C_1 \cdot C_2 \geq 2$. Hence

$$(D')^2 = \sum_{i=1}^k C_i^2 + 2 \sum_{1 \leq i < j \leq k} C_i \cdot C_j \geq 0.$$

This contradicts the Hodge index theorem, which asserts that for any divisor $L \in \text{Pic}(Y_0)$,

$$(K_{Y_0} \cdot L)^2 \geq K_{Y_0}^2 \cdot L^2,$$

and the equality holds if and only if $K_{Y_0} \sim_{\text{num}} rL$ for some rational number r . Hence D has at worst simple double points as its singularities.

Let δ be the number of its singularities. If $\delta = 0$, i.e., $D = C_1$ is irreducible, then it is clear that the inverse image of D is a (-1) -curve, which contradicts the fact that $X = S_t$ is minimal. Hence $\delta > 0$. Since D is a connected component of the branch divisor of a double cover, $(D - C_i) \cdot C_i \geq 2$ is even for any $1 \leq i \leq l$. It follows that $D \cdot C_i = C_i^2 + (D - C_i) \cdot C_i \geq 0$, which implies that $D^2 = D \cdot \sum_{i=1}^l C_i \geq 0$.

This is also a contradiction to the Hodge index theorem. The proof is complete.

(2) If $K_{Y_0}^2 \geq \frac{9}{2}\chi(\mathcal{O}_{Y_0})$, then by (4-3) and (4-4) together with the assumption that $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$ one obtains

$$\begin{aligned}
0 &> K_X^2 - \frac{9}{2}\chi(\mathcal{O}_X) \geq K_{S_t}^2 - \frac{9}{2}\chi(\mathcal{O}_{S_t}) \\
&= 2(K_{Y_0}^2 - 4\chi(\mathcal{O}_{Y_0})) + \frac{7}{8}K_{Y_0} \cdot R_0 - \frac{1}{16}R_0^2 + \frac{1}{4} \sum_{i=0}^{t-1} (m_i - 1)(m_i + 8) \\
&\geq \frac{7}{8}K_{Y_0} \cdot R_0 - \frac{1}{16}R_0^2.
\end{aligned}$$

Hence $R_0^2 > 14K_{Y_0} \cdot R_0$. On the other hand, according to the Hodge index theorem, one has $(K_{Y_0} \cdot R_0)^2 \geq K_{Y_0}^2 \cdot R_0^2$. Thus

$$K_{Y_0} \cdot R_0 > 14K_{Y_0}^2.$$

Therefore

$$\begin{aligned} K_X^2 - 4\chi(\mathcal{O}_X) &\geq K_{S_t}^2 - 4\chi(\mathcal{O}_{S_t}) \\ &= 2(K_{Y_0}^2 - 4\chi(\mathcal{O}_{Y_0})) + K_{Y_0} \cdot R_0 + 2 \sum_{i=0}^{t-1} (m_i - 1) \\ &> \chi(\mathcal{O}_{Y_0}) + 14K_{Y_0}^2 \geq 64\chi(\mathcal{O}_{Y_0}). \end{aligned}$$

Since Y_0 is of general type, one has the inequality $h^0(X, K_X) \geq 2q - 4$ (cf. [Bea83, Corollary X.8]). In other word, we have

$$\chi(\mathcal{O}_{Y_0}) \geq \min \{1, q - 3\}.$$

Hence

$$K_X^2 - 4\chi(\mathcal{O}_X) > 64 \cdot \min \{1, q - 3\} > 4(q - 2). \quad \square$$

To deal with the case when $q = 2$, we also need the following lemma.

Lemma 4.5. *Let $\tilde{\zeta} : \tilde{W} \rightarrow Z$ be a flat double cover of an abelian surface Z whose branch divisor $R \subseteq Z$ has at worst simple singularities, and $W \rightarrow \tilde{W}$ be the resolution of singularities. If W is of general type, then R is ample and we have the following isomorphism between the fundamental groups:*

$$\zeta_* : \pi_1(W) \xrightarrow{\cong} \pi_1(Z) \cong \mathbb{Z}^{\oplus 4}, \quad \text{where } \zeta : W \rightarrow Z \text{ is the composition.} \quad (4-8)$$

In particular, any unramified cover $\tau : W' \rightarrow W$ is induced from a unramified cover $v : Z' \rightarrow Z$ with the following commutative diagram

$$\begin{array}{ccc} W' & \xrightarrow{\tau} & W \\ \zeta' \downarrow & & \downarrow \zeta \\ Z' & \xrightarrow{v} & Z \end{array}$$

Proof. First we show that R is ample. According to (4-3) together with our assumption, we have $0 < K_Z^2 = \frac{1}{2}R^2$, from which it follows that $R^2 > 0$. For any curve $C \subseteq Z$, $R \cdot C \geq 0$ since Z is an abelian surface; otherwise, $C^2 < 0$ according to the Hodge index theorem, which is impossible as C is contained in an abelian surface. This shows that $R \cdot C > 0$. Hence R is ample.

Next we prove the isomorphism (4-8). Since R is the branch divisor of the double cover $\tilde{\zeta}$, there exists a line bundle \mathcal{L} such that $\mathcal{O}_Z(R) \equiv \mathcal{L}^{\otimes 2}$. Consider the complete linear system $|R|$ as a projective space and let $\Lambda \subseteq |R|$ be the open and dense subvariety corresponding to reduced divisors with at worst simple singularities. For each $\lambda \in \Lambda$, let $R_\lambda \subseteq |R|$ be the corresponding divisor and $\tilde{\zeta}_\lambda : \tilde{W}_\lambda \rightarrow Z$ the double cover determined by the relation $\mathcal{O}_Z(R_\lambda) \equiv \mathcal{L}^{\otimes 2}$. Then we obtain a flat family of projective surfaces

$$\tilde{f} : \bigcup_{\lambda \in \Lambda} \tilde{W}_\lambda \longrightarrow \Lambda.$$

Moreover, by our assumption, each fibre of \tilde{f} is irreducible and reduced with at worst rational double points as its singularities. Hence every fibre of \tilde{f} has the same fundamental group by [Xia91, Lemma 9]. On the other hand, since $\mathcal{O}_Z(R) \equiv \mathcal{L}^{\otimes 2}$, it follows that the complete linear system $|R|$ is base-point-free. Thus one can choose an element $R_\lambda \subseteq |R|$ such that R_λ is smooth. Hence according to [Nor83, Corollary 2.7], one has the following isomorphism:

$$(\tilde{\zeta}_\lambda)_* : \pi_1(\widetilde{W}_\lambda) \longrightarrow \pi_1(Z).$$

So we obtain an isomorphism

$$\tilde{\zeta}_* : \pi_1(\widetilde{W}) \longrightarrow \pi_1(Z).$$

Finally, Since \widetilde{W} has at worst rational double points as its singularities, one has a natural isomorphism $\pi_1(W) \rightarrow \pi_1(\widetilde{W})$. The proof is complete. \square

Proof of Theorem 1.3. (1). It suffices to prove that if $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$, then

$$K_X^2 \geq 4\chi(\mathcal{O}_X) + 4(q - 2). \quad (4-9)$$

We prove (4-9) by induction on the degree of the Albanese map of the surface. According to Theorem 3.1, $\deg(\text{Alb}_X) > 1$. More precisely, the Albanese map Alb_X of X factors through a double cover $\pi : X \rightarrow Y = X/\langle\sigma\rangle$. Let Y_0 be the minimal smooth model of Y . It follows from Lemma 4.1 that Y_0 is minimal surface of maximal Albanese dimension with

$$q(Y_0) = q, \quad \deg(\text{Alb}_{Y_0}) = \frac{1}{2} \deg(\text{Alb}_X).$$

Moreover, according to Proposition 4.2 and Lemma 4.4(2), we may assume that

$$K_{Y_0}^2 < \frac{9}{2}\chi(\mathcal{O}_{Y_0}).$$

In other word, Y_0 is a minimal surface of general type and maximal Albanese dimension with $K_{Y_0}^2 < \frac{9}{2}\chi(\mathcal{O}_{Y_0})$ and $\deg(\text{Alb}_{Y_0}) = \frac{1}{2} \deg(\text{Alb}_X)$. By induction, we obtain

$$K_{Y_0}^2 \geq 4\chi(\mathcal{O}_{Y_0}) + 4(q(Y_0) - 2).$$

Hence (4-9) follows from Lemma 4.4(1).

(2). We use the same notations as above. First it is clear that the equality (1-3) holds if the canonical model of X satisfies the condition in the theorem. Conversely, if (1-3) holds, then the Albanese map Alb_X of X factors through a double cover π by Theorem 3.1 since $K_X^2 < \frac{9}{2}\chi(\mathcal{O}_X)$. According to the proof of (1) together with Proposition 4.2 and Lemma 4.4, our theorem is proved unless $q = 2$, $\pi = \pi_0$ is unramified and $Y = X/\langle\sigma\rangle$ is minimal of general type of maximal dimension with

$$q(Y) = q, \quad K_Y^2 = 4\chi(\mathcal{O}_Y).$$

By repeating the above process, we obtain that if the equality (1-3) holds, then either the canonical model of X is a flat double cover of Y as required in our theorem, or $q = 2$ and the Albanese map Alb_X factors as

$$Z_0 = X \xrightarrow{\zeta_1 = \pi} Z_1 = Y \xrightarrow{\zeta_2} \cdots \xrightarrow{\zeta_{s-1}} Z_{s-1} \xrightarrow{\zeta_s} Z_s = \text{Alb}_X(X),$$

Alb_X

where $s \geq 2$, ζ_i is a unramified double cover for any $1 \leq i \leq s-1$, Z_{s-1} is of general type, Z_s is an abelian surface, and ζ_s is the resolution of a flat double

cover $\tilde{\zeta}_{s-1} : \tilde{Z}_{s-1} \rightarrow Z_s$ whose branch divisor has at worst simple singularities. To complete the proof, it suffices to derive a contradiction if the later possibility happens.

Assume that the later case happens. Then Z_i is of maximal Albanese dimension with

$$\deg(\text{Alb}_{Z_i}) = 2^{s-i}, \quad \forall 0 \leq i \leq s. \quad (4-10)$$

By Lemma 4.5, there exists an abelian surface Z' and an unramified double cover $v : Z' \rightarrow Z_s$ with the following commutative diagram.

$$\begin{array}{ccc} Z_{s-2} & \xrightarrow{\zeta_{s-1}} & W \\ \zeta' \downarrow & & \downarrow \zeta_s \\ Z' & \xrightarrow{v} & Z_s \end{array}$$

Since Z' is an abelian surface and $\deg(\zeta') = 2$, it follows that $\deg(\text{Alb}_{Z_{s-2}})$ is at most 2. It is a contradiction to (4-10). This completes the proof. \square

Proof of Corollary 1.4. It follows directly from Theorem 1.3 except the ampleness of the branch divisor in characterizing the equality $K_X^2 = 4\chi(\mathcal{O}_X)$, which comes from Lemma 4.5. \square

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